Exam
Mechanics & Relativity 2016–2017 (part Ib)
January 27, 2017

INSTRUCTIONS

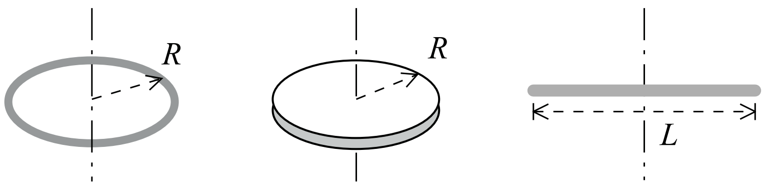
- Use a black or blue pen.
- Write the name of your tutor and/or group on the top right-hand corner of the first sheet handed in, and *deposit your work in the box with your tutor's name.*
- This exam comprises 5 problems. The first four problems require clear arguments and derivations, all written in a well-readable manner, the last one is a multiple choice question.
- The number of points for every subquestion is indicated inside a box in the margin. The total number of points per problem is

Problem	# of points
1	7
2	9
3	5
4	2
5	2

and the grade is computed as (total # points) $/25 * 9 + 1$.

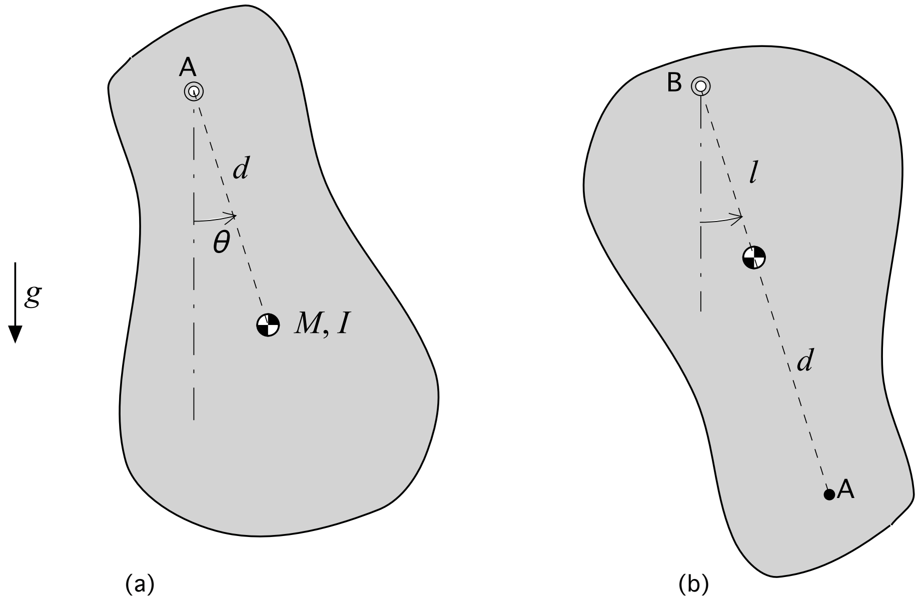
Some useful formulas

The mass of the ring, the thin disk and the rod shown below is M and distributed uniformly. The moments of inertia about the axis (shown by the dashed-dotted line) through their center of mass are:



$I = MR^2$ $\qquad$ $I = \frac{1}{2}MR^2$ $\qquad$ $I = \frac{1}{12}ML^2$

Problem 1 Every physical pendulum has an "anti" version, termed the inverted pendulum. Prior to studying its properties, we first analyse the "ordinary" pendulum in figure (a). The body, with mass M and moment of inertia I around the center of mass (CM) can rotate freely about the point A located at a distance d above the CM.



(a) (b)

a. Determine the potential energy V of gravity at an arbitrary value of θ (note that the mass of the object can be considered to be concentrated in the center of mass). [1]

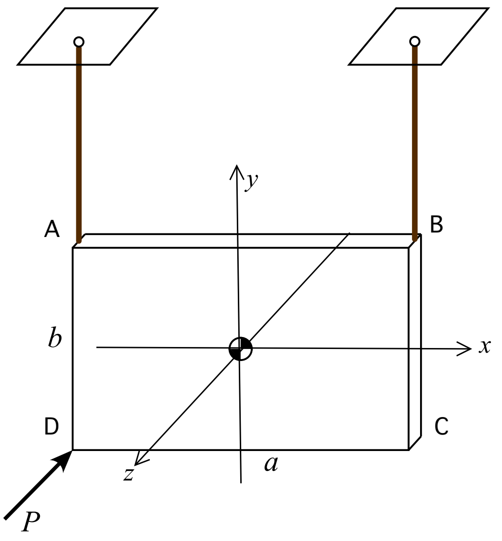
b. The kinetic energy T can be determined in two ways: 1) by determining the translational kinetic energy of, and the rotational kinetic energy around the center of mass; 2) by determining the rotational kinetic energy around the pivot A. Calculate the kinetic energy in both ways, and show that they give the same result, $T = \frac{1}{2} I_A \dot{\theta}^2$. [3]

c. Use conservation of energy to derive the harmonic equation for small oscillations about the equilibrium position (indicated with the vertical dash-dotted line). Show that that the eigenfrequency of these oscillations is given by $\sqrt{Mgd/I_A}$. [2]

Now, consider the situation in figure (b), where the same body can rotate about the point B located *opposite* to the CM at a distance l. The eigenfrequency of this "anti" pendulum will in general be different from the original one; in the special case that the frequencies are the same, the pendulum in figure (b) is called the inverted pendulum.

d. Determine the distance l for an inverted pendulum. [1]

Problem 2 A rectangular thin plate with uniform mass m, width a and height b, is suspended from two massless strings attached to the corners A and B. From a stationary situation, with the edge AB being horizontal, the corner D is kicked by an impulse ΔP perpendicular to the plate in the $-z$ direction in the figure below.



a. Determine the moments of inertia about the x, y and z axes.

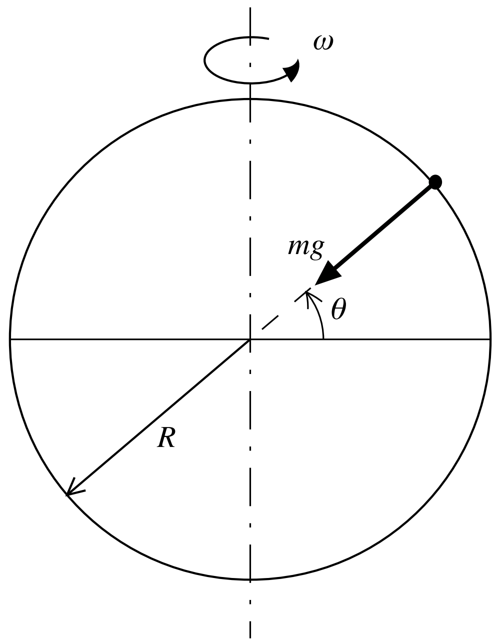
b. Compute the velocity of the center of mass immediately after the impulse,

c. as well as the spin (or, angular velocity) vector of the plate.

d. Determine the percussion line, i.e. the line on the plate that does not move after the blow.

2,1,2,4

Problem 3 (5 points) Treating the earth as a sphere (radius R), an object with mass m at the earth's surface is subjected to a gravitational force of magnitude mg in the direction of the center of the earth. This, however, is not the "g"-force that the object actually feels. Since the earth rotates about its own axis (angular velocity ω), the object seems to be subjected to an "effective" gravitational force that is not directed towards the center and whose magnitude depends on the latitude θ.

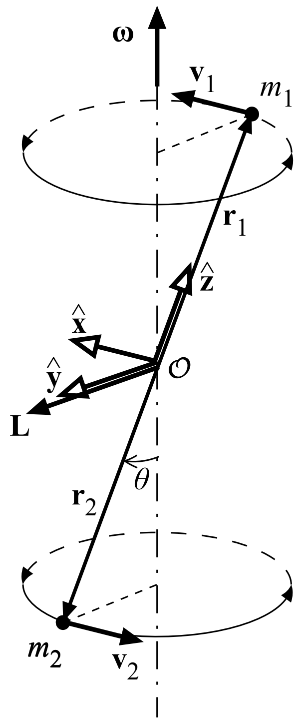


a. Which force other than gravity is acting on a stationary object? Determine the magnitude of this force as a function of θ. 2

b. Copy the figure above and add the direction of the total, "effective" gravitational force mg_{eff} to your sketch. 1

c. What other force(s) is the object subjected to when it moves towards the equator ($\theta = 0$)? Add its/their direction to the drawing under (b). 2

Problem 4 Consider a dumbbell consisting of a massless rod (length $2b$) with masses m_1 and m_2 at either end. The center of the dumbbell $\mathcal{O}$ is attached to an axis about which it rotates with angular velocity ω. The angle between dumbbell and rotation axis is θ. Attach a set of right-handed orthonormal base vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ to the rotating dumbbell with $\hat{\mathbf{z}}$ along the rod, and with $\hat{\mathbf{y}}$ in the direction of the angular momentum. It can be shown that the components of the angular momentum relative to this basis are given



2

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} 0 \\ (m_1 + m_2) b^2 \omega \sin\theta \\ 0 \end{pmatrix}$$

Consider the case of constant spin, i.e., $\dot{\omega} = 0$. Does $\boldsymbol{\tau} = d\mathbf{L}/dt$ imply that a torque is needed to sustain this motion or not? Motivate your answer.

Problem 5 (Multiple choice question: only give answer (A thru C), not reason why) The precession frequency of a top spinning at constant velocity:

A) is higher on earth than on the moon.

B) is lower on earth than on the moon.

C) is insensitive to gravity.

2

✍ ——— ✍

Answers
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Problem 1

a. $V = Mgd(1 - \cos\theta)$. 1

b. $T = \frac{1}{2} I_A \dot{\theta}^2$, where $I_A = I + Md^2$ is the moment of inertia relative to the pivot A. 3

c. $I_A \ddot{\theta} + Mgd \sin\theta = 0$, which for small θ gives the harmonic equation with eigenfrequency $\sqrt{Mgd/I_A}$ with $I_A = I + Md^2$. 2

d. $l = d$. 1

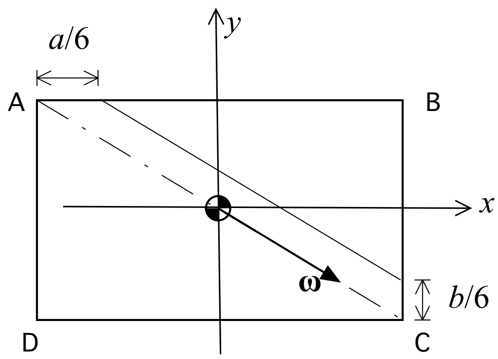
Problem 2

a. $I_x = \frac{1}{12} mb^2$; $I_y = \frac{1}{12} ma^2$; $I_z = I_x + I_y$. 2

b. $V_x = V_y = 0$, $V_z = -\Delta P/m$. 1

c. $\omega_x = 6\Delta P/(mb)$, $\omega_y = -6\Delta P/(ma)$, $\omega_z = 0$. 2

d. The line parallel to $\boldsymbol{\omega}$ but $a/6$ to the right (and, hence, $b/6$ up) does not move.



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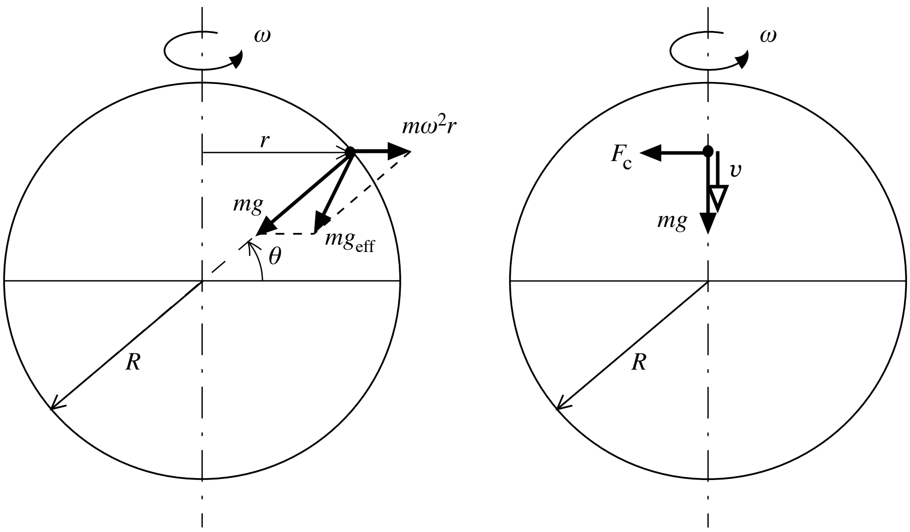
Problem 3

a. A centrifugal force perpendicular to the North-South axis of the earth, pointing outward and with a magnitude equal to $m\omega^2 r$ with $r = R\cos\theta$. 2

b. See diagram below, left-hand side.



1

c. Moving objects also seem to be subject to a Coriolis force F_c which deflect their motion to the right. For an object moving towards the equator, at velocity v, the Coriolis force points to the West. See above diagram, right-hand side. 2

Problem 4 Each of the components of **L** on this basis is constant when $\dot{\omega} = 0$, but the basis itself is not constant. Hence **L** is not constant. 2

Problem 5
A (the precession frequency is determined by the torque delivered by gravity; since the latter is larger on earth than on the moon, the precession frequency is higher on earth). 2